On the invariance of the relative rest in doubly special relativity

Gianluca Mandanici[*]

Università degli Studi di Bergamo, Facoltà di Ingegneria,
Viale Marconi 5, 24044 Dalmine (Bergamo) Italy.

In the framework of the most-studied doubly special relativity models the use of the naive formula $v = dE/dp$ has been argued to lead to inconsistencies connected to different rules of transformation, under boosts, of particles with the same energy but with different masses. In this paper we show that, at least in 1+1 dimensions, doubly special relativity can be formulated in such a way that the formula $v = dE/dp$ is fully consistent with the invariance of the relative rest, easily fitting to the relativity principle. It is also argued that, always in 1+1d, is not necessary to renounce to the usual (commutative) Minkowski space-time endowed with energy-independent boost transformations. The compatibility of the approach with superluminal propagation, with linear addition rule for energy, and possible extensions to 3+1 dimensions are also discussed.

I. INTRODUCTION

In the quest for a quantum theory of gravity notable interest has been attracted over the last decade and more, from scenarios in which the Lorentz/Poincaré symmetry is deformed (but not broken) at the Planck scale. A much studied class of these scenarios is often referred in literature as doubly special relativity [1–5] and typically consists in a two-scale extension of Einstein's special relativity, being the second invariant scale eventually connected with the Planck energy/length. At first, doubly special relativity has been formulated in the energy-momentum sector, however, over the years, many attempts have been made to consistently extend the theory from the energy-momentum sector to the space-time sector (for a recent review see [6]). A number of these attempts are based on the assumption of a underling non-commutative space-time as a consequence of the fuzziness of space-time at length scale of the order of the Planck length (see e.g. [7–11]). Other attempts instead assume a underlying commutative space-time (see e.g. [12, 13]). What is common within most of these approaches is that one obtains energy-dependent space-time transformations with the consequence that particles with the same speed, but with different masses, acquire relative motion when considered from different inertial observers. This unclassical feature of space-time transformations could appear in contrast with the relativity principle especially for the apparent loss of locality [14], that is recovered only in a form that depends on the choice of the inertial frame [15–17]. A central role in the delicate transition from the energy-momentum sector to the space-time sector of doubly special relativity is played by the formula that expresses the speed of a free particle as a function of its energy and its momentum. While the first proposals [1–3] have assumed the validity of the usual formula $v = dE/dp$, in [18] this formula been argued to be incompatible with the standard hamiltonian formalism. In [19] starting from the request that the formula of the particle speed should not depend on the mass of the particles, the author was led to rule out the formula $v = dE/dp$. Similar results have been obtained in [20] where has also been claimed that the dependence of the velocity on the mass of the boosted particle appear unavoidable using the mentioned formula. A study of velocity of particle in non-commutative space-time still compatible with the formula $v = dE/dp$ is instead that of [21]. The main aim of this paper is just to address the issue of the relative rest in doubly special relativity and its compatibility with the standard formula $v = dE/dp$. The structure of the paper is the following. In section II we discuss the implications of the request of covariance of the velocity in the form dE/dp for the energy-momentum sector of the most-studied doubly special relativity models, providing explicit examples of covariance. In section III we propose a construction of the space-time sector from given energy-momentum sectors. In section IV we address the issues of the compatibility of the approach with deformations of the boost action on the particle speed, with linear addition rule for energy and we also discuss a possible extension to 3+1 space-time dimensions. Finally, in section V, we present our conclusions.

---

[*]Electronic address: Gianluca.Mandanici@unibg.it

II. MOMENTUM SPACE FORMULATION

A. The classical relativistic picture of relative rest

The first working assumption of this paper is that *particles that do not possess relative motion in a given reference frame do not acquire relative motion in any other (inertial) reference frame.* In the usual formulation, for infinitesimal transformations, the speed of a boosted particle v' is obtained from the speed of the particle v by means of the formula

$$v'(\xi) \simeq v + \xi \{N, v\}, \tag{1}$$

where ξ is the boosting parameter and $\{N, v\}$ is the infinitesimal boost action. In this notation, in order to obtain the invariance of the relative rest, the boost operator must be function only of the particle speed v and of (possible) fundamental constants $c_1, ..., c_n$ of the model

$$\{N, v\} = K(v, c_1, ..., c_n). \tag{2}$$

Notice that this request is satisfied in Galilei relativity where no invariant velocity scale is present. In fact in Galilei relativity, being $E = p^2/(2m)$, $v = dE/dp = p/m$ and $\{N, p\} = m$, one finds

$$\{N, v\} = \{N, p\} \frac{dv}{dp} = 1, \tag{3}$$

so that $K(v) = 1$. Also in Einstein's special relativity, where the invariant velocity scale c is present, being $E^2 = c^2 p^2 + c^4 m^2$, $v = dE/dp = c^2 p/E$ and $\{N, p\} = E/c^2$, one finds

$$\{N, v\} = \{N, p\} \frac{dv}{dp} = 1 - c^2 p^2/E^2, \tag{4}$$

i.e. $K(v, c) = 1 - v^2/c^2$. Again the function $K(v)$ depends only on the particle speed and on the invariant scale c. In this latter case one also gets $K(c, c) = 0$ consistently with the fact that c is not only an invariant parameter of the theory but, rather, it is the invariant speed of (massless) particles. In both these known cases $K(v)$ depends only on the particle speed, so that two particles with the same speed in a given reference frame $v_1 = v_2 = v$ do not acquire relative motion in any other inertial reference frame

$$v'_2 - v'_1 \simeq \xi[K(v, c) - K(v, c)] = 0, \tag{5}$$

independently on their masses.

B. Relative rest in the most studied doubly special relativity models

In most approaches to Planck-scale relativity there are strong departure from the classical picture as outlined above. In fact if one adopts the formula[1]

$$v = \frac{dE}{dp}, \tag{6}$$

one finds that in general $K = K(v, c, \lambda c m, \lambda p)$, where λ is the Planck-scale parameter, eventually identified with the inverse of the Planck momentum. This means that the action of the boosts on the speed of a particle depends not only on the particle speed itself but also on the particle mass. The main consequence of this dependency is that particles having the same speeds $\overline{v}$ in a given reference frame could acquire relative motion in a different reference frame, if they have different masses. In fact, being in general $K(\overline{v}, c, \lambda c m_1, \lambda p_1) \neq K(\overline{v}, c, \lambda c m_2, \lambda p_2)$, one finds

$$\overline{v}'_1 - \overline{v}'_2 = \xi[K(\overline{v}, c, \lambda c m_1, \lambda p_1) - K(\overline{v}, c, \lambda c m_2, \lambda p_2)] \neq 0. \tag{7}$$

---

[1] We emphasize here that this formula that works well in classical physics and quantum mechanics can also be derived in any Hamiltonian approach $v = \dot{x} = \{x, E(p)\}$ also compatible with $\{x, p\} = 1$, being $v = \{x, E\} = \{x, p\} dE/dp = dE/dp$.

This effect has been analyzed in [20] in the framework of the so-called DSR1 model proposed in [1, 2] that admits energy-momentum dispersion relation of the type

$$E = c \, |\lambda|^{-1} \ln \left[\frac{\cosh(\lambda mc) + \sqrt{\cosh^2(\lambda mc) - (1 - \lambda^2 p^2)}}{(1 - \lambda^2 p^2)} \right], \tag{8}$$

but the same effect is also present in the other most-studied doubly special relativity model, the so-called DSR2 model, proposed in [3]. Again can be easily shown that from energy-momentum dispersion relation of the form

$$\frac{E^2 - p^2 c^2}{(1 - \lambda E/c)^2} = \frac{m^2 c^4}{(1 - \lambda mc)^2}, \tag{9}$$

using (2) and (6) one finds

$$K(v, c, \lambda m) = \frac{1 - v^2/c^2}{1 + \lambda cm / \sqrt{(1 - \lambda cm)^2 - v^2(1 - 2\lambda cm)/c^2}}. \tag{10}$$

From eq.(10) follows that two particles at rest in a given reference frame ($v_1 = v_2 = 0$), will acquire a relative motion:

$$v_1' - v_2' = \xi[K(0, c, \lambda m_1) - K(0, c, \lambda m_2)] \simeq \xi \lambda c (m_1 - m_2) \neq 0. \tag{11}$$

The dependency of the boosts on the mass of the particle does not necessarily imply a violation of the relativity principle but, at least, suggests that locality should become a concept relative to the particular choice of the reference frame (see e.g. [15–17]). In the rest of this paper we wont address the point of whether or not effects of the type implied by formula (11) can be incorporated in a relativistic scheme. Rather we intend to focus here on what could be the features of doubly special relativity models in which effects implied by formula (11) are absent at all, like they are in the known classical cases.

C. Approaching relative rest invariance in doubly special relativity

1. General theoretical framework

Once we have discussed the way in which the problem manifests in doubly special relativity we can move forward looking for a solution. We start requiring the covariance under boosts of the energy-momentum dispersion relation $E = E(p)$. From this request it follows that it must be

$$\{N, E\} = \{N, p\} \frac{dE}{dp}, \tag{12}$$

that fixes the relation between the action of the boosts on the energy and the action of the boosts on the momentum. Our next request is that $\{N, v\}$ be a function of only the particle speed, expressed as in eq.(6), from which one gets the following formula

$$\{N, v\} = \{N, p\} \frac{d^2 E}{dp^2} = K \left(\frac{dE}{dp} \right). \tag{13}$$

The validity of eq.(13) directly implies that particles with the same speed (i.e. at rest in a given reference frame) wont acquire any relative motion in any other reference frame. Thus the request that the boosted velocity depends only on the particle speed fixes the infinitesimal action of the boost on the momentum in the form:

$$\{N, p\} = K \left(\frac{dE}{dp} \right) / \frac{d^2 E}{dp^2}. \tag{14}$$

Finally substituting (14) in (12) one obtains the infinitesimal action of the boost on the particle energy

$$\{N, E\} = K \left(\frac{dE}{dp} \right) \frac{dE}{dp} / \frac{d^2 E}{dp^2}. \tag{15}$$

Formulas (14)-(15) together with (6) are the main working tools of the analysis presented in the rest of the paper.

2. A first-order example in the Planck scale

We are now ready to apply the procedure outlined above to a general Planck-scale deformed dispersion relation. As a warm up we start the analysis with a first order deformation. The dispersion relation that we consider here is of the type of those studied in [22, 23], that is a most studied energy-momentum dispersion relation:

$$E^2 - c^2 p^2 + \lambda p^3 c^2 = m^2 c^4. \tag{16}$$

Particle speed can be calculated from the dispersion relation using eq.(6) obtaining

$$v = \frac{dE}{dp} = \frac{c^2 p(1 - 3/2\lambda p)}{\sqrt{c^4 m^2 + p^2 c^2 - \lambda c^2 p^3}}, \tag{17}$$

that, for massless particles, assumes the form $v \simeq c(1 - \lambda p)$, at the first order in λ. To derive the action of the boosts on the energy and on the momentum we use formulas (14)-(15). The form of $K(v)$ is in principle arbitrary unless it provides the correct Minkowski limit. However, in this section, we will assume an undeformed expression for $K(v)$. Thus fixing $K(v) = 1 - v^2/c^2$, one finds for the action of the boost:

$$\{N, p\} = \frac{\left[4c^2 m^2 + p^3 \lambda (8 - 9p\lambda)\right] \sqrt{c^2 m^2 + p^2 - \lambda p^3}}{cp^3 \lambda(-4 + 3p\lambda) + 4c^3 m^2 (1 - 3\lambda p)}, \tag{18}$$

$$\{N, E\} = \{N, p\} \frac{c^2 p(1 - 3/2\lambda p)}{\sqrt{c^4 m^2 + p^2 c^2 - \lambda c^2 p^3}}, \tag{19}$$

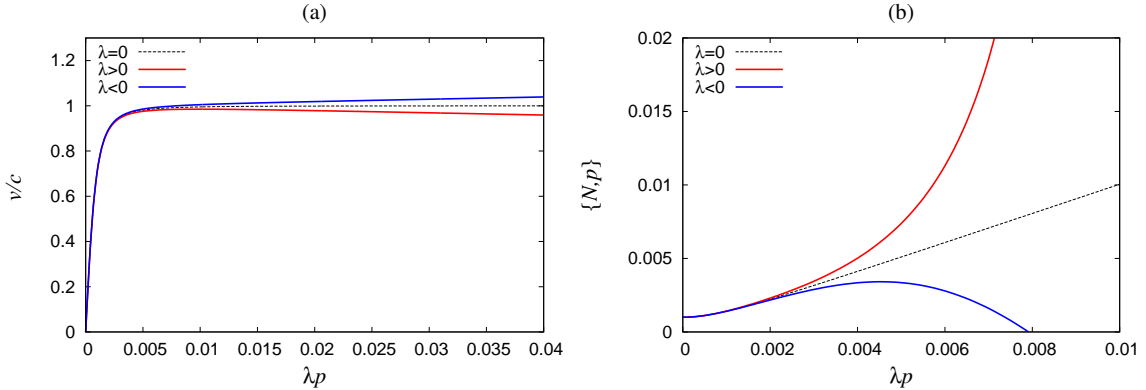whose behavior, as a function of the particle momentum p, is reported in Fig. 1.

Figure 1: (a) v/c vs λp and (b) $\{N, p\}$ vs λp for the dispersion relation of eq.(16) with $mc\lambda = 0.001$. The black-dashed lines ($\lambda = 0$) are the predictions of special relativity.

It is easy to check that $i)$ the dispersion relation (16) is invariant under boost action and that $ii)$ we have $v_1' - v_2' \simeq \xi \left[K(0, c, \lambda cm_1) - K(0, c, \lambda cm_2)\right] = 0$ for all the possible values of the masses m_1 and m_2 of the particles involved. Thus the state of relative quite is preserved for all the particles in all the inertial frames. The entire analysis here reported is performed in the energy-momentum space. The only connection with space-time is provided by formula (6) that defines the particle speed. In the case of $\lambda = 0$ one recovers the usual special relativistic formulas. The case $\lambda < 0$, that in principle should correspond to superluminal propagation (i.e. to $v > c$), in our approach returns $\{N, p\} \simeq \{N, E\} \simeq 0$ as soon as $p \simeq p_I = \sqrt[3]{m^2 c^3 \lambda^{-1}/2}$, as it is clear both from the analytical expressions (18)-(19) and from Fig. 1. This means that p_I is an invariant momentum ($p_I' \simeq p_I$), and the corresponding energy an invariant energy ($E_I' \simeq E_I$). The reasons why one gets invariant energy and momentum from eqs. (18)-(19) is that as soon as the momentum approaches p_I the speed of the particle involved approaches the invariant speed c, being in fact $\{N, v(p_I)\} \simeq 0$. Notice that the invariant momentum p_I depends on the mass of the particle involved and that it is different from the Planck momentum, as far as the mass of the particle involved is different from the Planck mass. The case $\lambda > 0$ corresponds to subluminal propagation (i.e. to $v < c$). In this case the particle never approaches $v \simeq c$ and thus there is no invariant momentum. However there is a maximum momentum since, as soon as $p \simeq p_{max} = \sqrt[3]{m^2 c^3 \lambda^{-1}}$, one gets $\{N, p_{max}\} \simeq \{N, E(p_{max})\} \simeq \infty$.

3. Two all-order examples: the Magueijo-Smolin dispersion relation and the κ-Poincaré dispersion relation.

We are now ready to apply our procedure to all-order (in the Planck scale) dispersion relations. The first type of dispersion relation that we consider is the Magueijo-Smolin dispersion relation (9). Again using formula (6) one gets the particle speed:

$$v = \frac{dE}{dp} = \frac{c^2 p (1 - mc\lambda)^2}{(1 - 2\lambda cm) E + m^2 c^3 \lambda}. \tag{20}$$

From eqs.(14)-(15) we obtain that the transformation rules between inertial observers are given by the actions

$$\{N, p\} = \frac{E(1 - 2mc\lambda) + m^2 c^3 \lambda}{c^2 (1 - mc\lambda)^2} (1 - \lambda^2 p^2), \tag{21}$$

$$\{N, E\} = p(1 - \lambda^2 p^2), \tag{22}$$
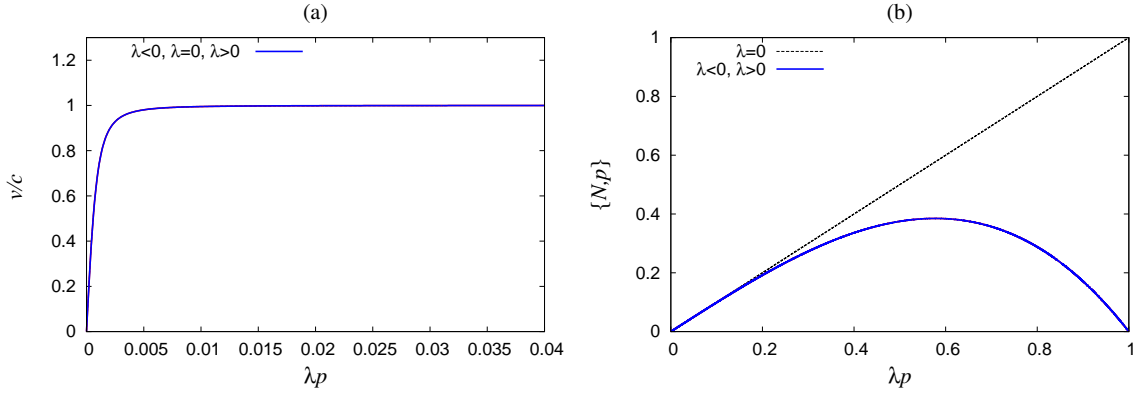
whose behavior is reported in Fig. 2.

Figure 2: (a) v/c vs λp and (b) $\{N, p\}$ vs λp for the dispersion relation of eq.(9) with $mc\lambda = 0.001$. The black-dashed line ($\lambda = 0$) is the prediction of special relativity.

As expected as v approaches the invariant speed c, the energy and the momentum approaches the invariant energy $c\lambda^{-1}$ and the invariant momentum λ^{-1} respectively, as in the case of the original boost actions discussed in [3] but now, differently from [3], it is easy to check that also

$$v_1 = v_2 \Rightarrow v_1' = v_2', \tag{23}$$

independently on the particle masses. Notice that it is also possible to express the energy and the momentum in terms of the mass and the speed of the particle

$$p(v) = \frac{mv}{\sqrt{(1 - \lambda mc)^2 - v^2 (1 - 2mc\lambda)/c^2}}, \tag{24}$$

$$E(v) = \frac{1}{1 - 2\lambda mc} \frac{m (1 - \lambda cm)^2 c^2}{\sqrt{(1 - \lambda mc)^2 - v^2 (1 - 2mc\lambda)/c^2}} - \frac{m^2 c^3 \lambda}{1 - 2\lambda cm}. \tag{25}$$

The second example of all-order dispersion relation that we want to consider here is the κ-Poincaré motivated dispersion relation of eq.(8). Following the same procedure we can easily find v, $\{N, p\}$ and $\{N, E\}$ (again involving eqs.(6), (14) and (15)). Being the analytical expressions rather involved we do not report them. Instead we represent graphically the results in Fig. 3 only for case $\lambda > 0$, since the case $\lambda < 0$ does not possess, in our approach, the proper classical limit (see also the discussion in [6]).
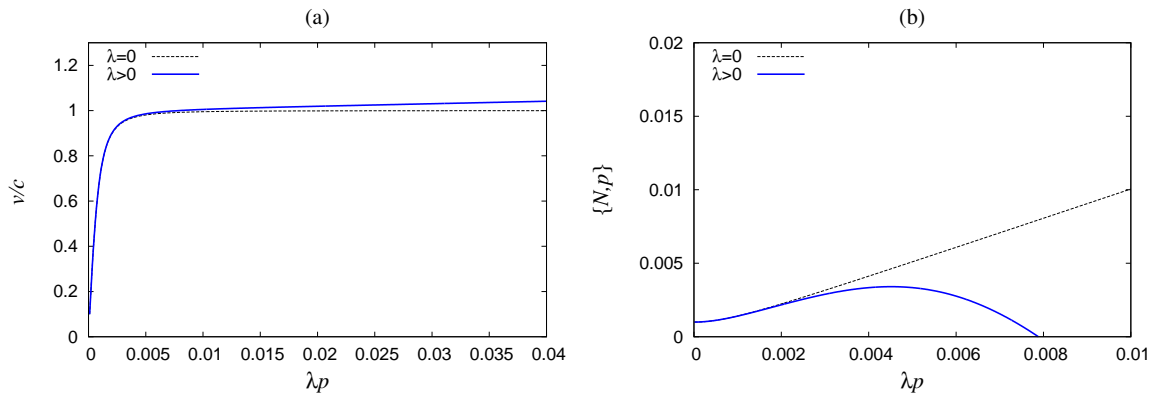
Figure 3: (a) v/c vs λp and (b) $\{N, p\}$ vs λp for the dispersion relation of eq.(8) with $mc\lambda = 0.001$. The black-dashed lines ($\lambda = 0$) are the predictions of special relativity.

III. THE SPACE-TIME SECTOR

In the preceding sections we have analyzed the invariance of the relative rest in the energy-momentum sector. In this section we try to address the same issue in the space-time sector. The link between the energy-momentum sector and the space-time sector is played by the definition of the particle speed given that, according to our assumptions, the following formula must hold

$$v = \frac{dE}{dp} = \frac{dx}{dt}. \tag{26}$$

The transformation rules in the energy-momentum sector have already been obtained therefore it is necessary that space-time transforms accordingly. We request that, as usual, different inertial systems are connected by uniform motion. Since we are looking for space-time boost transformations of the form

$$x'(\xi) \simeq x + \xi\{N, x\}, \tag{27}$$

in order to have inertial systems moving at a uniform relative motion it must be

$$\{N, x\} = t. \tag{28}$$

Then the covariance in the energy-momentum sector, together with the request of covariance of (26), fixes the transformation rule

$$\{N, dt\} = \left\{N, \frac{dx}{v}\right\} = \frac{1 - K(v)}{v}dt. \tag{29}$$

Notice that the transformation of the time under boost does not depend on the mass of the particles involved, it depends only on their speeds. Notice also that the action of boost on time can be a non-linear function of the particle speed v, depending on the particular form of $K(v)$. Since we do not want to introduce new invariant velocity scales, we will assume that K has not to be modified with respect to the standard special relativistic case.[2] Adopting the usual form $K(v) = 1 - c^{-2}v^2$ and substituting in (29) we get the action

$$\{N, t\} = c^{-2}x, \tag{30}$$

that is the usual infinitesimal action in the Minkowski space-time. It follows that all the elements regarding the space-time sector are not affected by the introduction of the Planck scale-parameter. In particular we get the usual

---

[2] The possibility of introducing a second velocity scale in this framework is not void of interest and can deserve further investigations.

dependence of the boost parameter ξ on the relative velocity v_S of the inertial systems, and the usual velocity composition rule

$$v_S = c \tanh\left(\frac{\xi}{c}\right), \qquad\qquad v' = \frac{v + v_S}{1 + \frac{vv_S}{c^2}}. \qquad (31)$$

Notice however that being in general $E(p) \neq E_{SR}(p)$ it follows that $p = (dE/dp)^{-1}(v) \neq (dE_{SR}/dp)^{-1}(v)$ so that the momentum and the energy depend on the speed of the particle in a deformed way with respect to the special relativistic case, as already evident from eqs.(24)-(25).

IV. FURTHER ISSUES

In this section we consider other issues related to the construction developed. The issues that we want to address are *i)* the possibility of adopting different forms of the function $K(v)$, *ii)* the compatibility of the formalism with a linear addition law for energies, *iii)* the possibility to extend the formalism from 1+1 dimensions to 3+1 dimensions.

A.   On the possibility of different $K(v)$

The possibility of modifying the usual Minkowski space-time means, in the framework of this paper, considering deviations from the special relativistic formula $K(v) = 1 - v^2/c^2$. Since our basic assumption is that $K(v)$ be a function only of $v = dE/dp$, we consider deformations of the type

$$K(v) = \sum_{k=0}^{n} a_{2k} \left(\frac{v}{c}\right)^{2k}, \qquad (32)$$

where $k = 0$, $a_0 = 1$ is the usual Galilean term and $k = 1$, $a_2 = -1$ is the special-relativistic correction. Notice that according to eq.(32) c is not necessarily an invariant velocity anymore, being in general $K(c) \neq 0$. However if we restrict our attention to function $K(v)$ such that

$$\sum_{k=0}^{n} a_{2k} = 0, \qquad (33)$$

then $K(c) = 0$, so that c is still an invariant speed. If now we substitute (32) in eqs.(14)-(15) and consider dispersion relations of the type of eq.(16), being

$$v = \frac{dE}{dp} \simeq c\left(1 - \frac{m^2 c^2}{2p^2} - \lambda p\right), \qquad (34)$$

$$\frac{d^2 E}{dp^2} \simeq c\left(\frac{m^2 c^2}{p^3} - \lambda\right), \qquad (35)$$

we get in the large momentum limit ($p \gg mc$ and $m \ll M_{Planck} = \lambda^{-1} c^{-1}$)

$$\{N, p\} \simeq \frac{\sum_{k=0}^{n} a_{2k}\left(1 - \frac{m^2 c^2}{2p^2} - \lambda p\right)^{2k}}{c\left(\frac{m^2 c^2}{p^3} - \lambda\right)}, \qquad\qquad \{N, E\} \simeq \frac{\sum_{k=0}^{n} a_{2k}\left(1 - \frac{m^2 c^2}{2p^2} - \lambda p\right)^{2k+1}}{c\left(\frac{m^2 c^2}{p^3} - \lambda\right)}. \qquad (36)$$

When $\lambda > 0$, that here means subluminal propagation, it is not possible for $\{N, p\}$ and $\{N, E\}$ to vanish. Thus we cannot have in this case neither an invariant momentum/energy nor an invariant speed of particle, not even for massless particles. Moreover as soon as the momentum reaches $p \sim \sqrt[3]{m^2 c^3 \lambda^{-1}}$ the denominators of eqs.(36) diverge. There is again a maximum allowed momentum $p_{\max} \sim \sqrt[3]{m^2 c^3 \lambda^{-1}}$ that depends on the particle mass and that can be many orders of magnitude smaller than the Planck momentum $|\lambda|^{-1}$. Instead in the case $\lambda < 0$ we get that the invariant speed c is reached as soon as the momentum becomes of the order of $p_I \sim \sqrt[3]{m^2 c^3 |\lambda|^{-1}/2}$.[3] For the same

---

[3] Notice that is this case the denominator of $\{N, p\}$ does not vanish so that $\{N, p\}$ does not diverge.

value of the momentum, the numerators of (36) vanish so that the momentum p_I is an invariant momentum and the corresponding energy an invariant energy. Thus the modifications introduced in $K(v)$ do not lead to significant differences in the predictions of the model. Always maintaining the same deformed form of $K(v)$ it is worth considering different dispersion relations. If we focus on the lowest orders, in the Planck parameter, of the dispersion relations of eq.(9) we get

$$v = \frac{dE}{dp} \simeq c \left(1 - \frac{m^2 c^2}{2p^2} - \frac{m^3 c^4}{p^2} \lambda + \frac{c^4 m^2}{2} \lambda^2 \right), \tag{37}$$

$$\frac{d^2 E}{dp^2} \simeq c \left(\frac{m^2 c^2}{p^3} + \frac{2m^3 c^4}{p^3} \lambda \right), \tag{38}$$

that suggest that $v = c$ is still the invariant speed for massless particles, independently on their momenta, and that $p \sim \lambda^{-1}$ remains an invariant momentum (also) for massive particles. Even considering a dispersion relation that provides the following

$$v = \frac{dE}{dp} \simeq c \left(1 - \frac{m^2 c^2}{2p^2} - \alpha p^2 m \lambda^3 \right), \tag{39}$$

$$\frac{d^2 E}{dp^2} \simeq c \left(\frac{m^2 c^2}{p^3} - 2\alpha pm \lambda^3 \right), \tag{40}$$

where α is a further deformation parameter, we obtain that, for $\alpha > 0$, the invariant speed would be reached at the invariant momentum $p_I \sim \sqrt[4]{mc^2/|\lambda|^3}$. Therefore, in all the analyzed cases, we find only minor departures from the behavior that the models exhibit with undeformed $K(v)$.

B. On the compatibility of the approach with linear energy addition rules

The second point that we want to address is the compatibility of our approach with the conservation of the energy for composite systems as analyzed in [24]. We have found so far that the covariance of the dispersion relation implies eq.(12) and that the covariance of the evolution equation $v = \{x, E\}$, together with $\{x, p\} = 1$, implies eq.(14). In order to guarantee covariance to the rule of additivity of the energy (the momentum remaining not additive) according to [24] it is enough to choose

$$\{N, E\} = \pi(p), \qquad \{N, p\} = E \left(\frac{d\pi}{dp} \right)^{-1}, \tag{41}$$

where $\pi(p)$ is a general function of the physical momentum p transforming according the usual special relativistic rules. From eq.(41) (see [24]) follows $E = \sqrt{m^2 + \pi^2}$. Thus the point reduces to find a dispersion relation compatible with eqs.(41) and with eqs.(12)-(14). In order to achieve this, one has to solve the following system

$$\begin{cases} E \left(\frac{d\pi}{dp} \right)^{-1} \frac{d^2 E}{dp^2} & = K \left(\frac{dE}{dp} \right) \\ \frac{dE}{dp} E \left(\frac{d\pi}{dp} \right)^{-1} & = \pi \end{cases} \tag{42}$$

always with $E = \sqrt{m^2 + \pi^2}$. Assuming again $K(v)$ to be undeformed, the second equation of (42) is trivially satisfied for every $\pi(p)$. Instead the first equation implies

$$\frac{d\pi}{dp} = 1 \Rightarrow \pi(p) = p. \tag{43}$$

Thus the only additive model compatible with our hypotheses, already in the 1+1-dimensional case, is the usual special-relativistic one.

C. On the possibility to extend the analysis to 3+1 dimensions

Finally let us came to the possibility to extended our analysis to 3+1 space-time dimensions. Particle speed in a boosted reference frame transforms according to the formula

$$v_i' = v_i + \xi_j \{N_j, p_k\} \frac{\partial^2 E}{\partial p_i \partial p_k}. \tag{44}$$

The request of covariance of the energy-momentum dispersion relation in 3+1 dimensions becomes

$$\{N_i, E\} = \frac{\partial E}{\partial p_k} \{N_i, p_k\}. \tag{45}$$

Then the request that the boosted velocity depends only on the particle speed implies, in the 3+1-dimensional case, the following expression for the infinitesimal action of the boost on the spatial momentum:

$$\{N_j, p_k\} = \left(\mathbf{H}^{-1}\mathbf{K}\right)_{jk}, \tag{46}$$

where we have defined

$$\mathbf{H}_{ik} = \frac{\partial^2 E}{\partial p_i \partial p_k}, \qquad \mathbf{K}_{ij} = K_{ij}\left(\frac{\partial E}{\partial p_i}, \frac{\partial E}{\partial p_j}\right). \tag{47}$$

For the space-time sector in 3+1 space-time dimensions, we get by means of the formula

$$v_i = \frac{\partial E}{\partial p_i} = \frac{dx_i}{dt}, \tag{48}$$

and following the procedure of the 1+1-dimensional case, the boost actions on the space-time coordinates:

$$\{N_j, x_j\} = \delta_{ij}t, \qquad \{N_i, dt\} = \left\{N_i, \frac{dx_j}{v_j}\right\} = \frac{\delta_{ij} - K_{ij}}{v_j}dt. \tag{49}$$

Again we can maintain the usual Minkowski commutative space-time adopting the standard 3+1-dimensional expression $\mathbf{H}_{ik} = c^2 E^{-1}\left(\delta_{ij} - p_i p_j c^2 E^{-2}\right)$ and $\mathbf{K}_{ij} = \delta_{ij} - 1/c^2 \partial E/\partial p_i \partial E/\partial p_j$. However, in this 3+1-dimensional case, it can be shown by explicit calculations that the transformation rules constructed using formulas (45)-(46) are not compatible with the group structure of the boost actions in the momentum space.

V. FINAL REMARKS AND CONCLUSIONS

Under the hypothesis that the action of the boosts on the speed of a particle be a function of the particle speed alone, i.e. that $\{N, v\} = K(v)$, we have addressed the issue of the relative rest in doubly special relativity. Our analysis has shown a way to construct, at least in 1+1 dimensions, double special relativity models in which the relative rest is not an inertial-observer dependent notion. The key point of our work is that our results can be obtained without renouncing to the usual $v = dE/dp$ formula for the particle speed. We have also argued that, in order our scheme to be fulfilled, there is no need to renounce to the usual (commutative) Minkowski space-time endowed with standard energy-independent boosts. From a conservative viewpoint we have mainly focused on the undeformed function $K(v) = 1 - v^2/c^2$. Within these assumptions we have found that those models in which particle speed approaches $v \simeq c$ admit invariant momentum p_I and invariant energy E_I that depend on the mass of the boosted particle. For instance, in the case of dispersion relation of the type $E^2 - c^2p^2 + \lambda p^3 c^2 = m^2 c^4$ and for dispersion relation of the type DSR1, the invariant momentum has the form $p_I \sim \sqrt[3]{m^2 c^2 \lambda^{-1}/2}$. Considering the available bounds on photon mass $m_\gamma \lesssim 10^{-18}eV/c^2$ and assuming $\lambda^{-1} \sim 10^{19}GeV/c$ we would get for photons an invariant momentum $p_I \lesssim 10^{-3}eV/c$, that would be ruled out by available data. Instead, in the case of the DSR2 dispersion relation, we find that remains $p_I \sim \lambda^{-1}$ independently on the mass of the particle considered. Maintaining a Minkowski structure for space-time and a standard form of $K(v)$ also implies that the transformation laws of angular frequency ω and wavenumber k must differ, in general, from those of E and p, with influences on the phenomenology. In fact one would expect no deviations from special relativity when considering effects such as the Doppler effect (in space-time coordinates) and/or the time dilatation between inertial observers. Instead, as discussed, deviations from special relativity manifest when considering quantities involving the energy-momentum sector. Different form of $K(v)$ can modify the phenomenology but, within the class of functions that we have analyzed, no qualitative changes have emerged with respect to the described picture. The action of boosts on the energy-momentum space appears however incompatible with linear addition rules for energies in the sense of [24]. We also have found troublesome to extend the analysis to 3+1-spatial dimensions especially because of difficulties in maintaining the group structure of the boost actions.

[1] G. Amelino-Camelia, "Testable scenario for relativity with minimum-length," *Phys. Lett.*, vol. B510, pp. 255–263, 2001.

[2] G. Amelino-Camelia, "Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale," *Int. J. Mod. Phys.*, vol. D11, pp. 35–60, 2002.

[3] J. Magueijo and L. Smolin, "Lorentz invariance with an invariant energy scale," *Phys. Rev. Lett.*, vol. 88, p. 190403, 2002.

[4] J. Kowalski-Glikman and S. Nowak, "Doubly special relativity theories as different bases of kappa Poincare algebra," *Phys.Lett.*, vol. B539, pp. 126–132, 2002.

[5] G. Amelino-Camelia, J. Kowalski-Glikman, G. Mandanici, and A. Procaccini, "Phenomenology of doubly special relativity," *Int.J.Mod.Phys.*, vol. A20, pp. 6007–6038, 2005.

[6] G. Amelino-Camelia, "Doubly-Special Relativity: Facts, Myths and Some Key Open Issues," *Symmetry*, vol. 2, pp. 230–271, 2010.

[7] J. Lukierski, H. Ruegg, and W. Zakrzewski, "Classical and Quantum Mechanics of Free k-Relativistic Systems," *Annals of Physics*, vol. 243, no. 1, pp. 90 – 116, 1995.

[8] J. Kowalski-Glikman and S. Nowak, "Noncommutative space-time of doubly special relativity theories," *Int.J.Mod.Phys.*, vol. D12, pp. 299–316, 2003.

[9] S. Mignemi, "Transformations of coordinates and Hamiltonian formalism in deformed special relativity," *Phys.Rev.*, vol. D68, p. 065029, 2003.

[10] P. Galan and G. A. Mena Marugan, "Canonical realizations of doubly special relativity," *Int. J. Mod. Phys.*, vol. D16, pp. 1133–1147, 2007.

[11] S. Ghosh and P. Pal, "Deformed Special Relativity and Deformed Symmetries in a Canonical Framework," *Phys.Rev.*, vol. D75, p. 105021, 2007.

[12] J. Magueijo and L. Smolin, "Generalized Lorentz invariance with an invariant energy scale," *Phys.Rev.*, vol. D67, p. 044017, 2003.

[13] D. Kimberly, J. Magueijo, and J. Medeiros, "Nonlinear relativity in position space," *Phys.Rev.*, vol. D70, p. 084007, 2004.

[14] S. Hossenfelder, "Bounds on an energy-dependent and observer-independent speed of light from violations of locality," *Phys.Rev.Lett.*, vol. 104, p. 140402, 2010.

[15] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, "The principle of relative locality," *Phys.Rev.*, vol. D84, p. 084010, 2011.

[16] G. Amelino-Camelia, M. Matassa, F. Mercati, and G. Rosati, "Taming Nonlocality in Theories with Planck-Scale Deformed Lorentz Symmetry," *Phys.Rev.Lett.*, vol. 106, p. 071301, 2011.

[17] G. Amelino-Camelia, L. Barcaroli, and N. Loret, "Modeling transverse relative locality," *Int.J.Theor.Phys.*, vol. 51, pp. 3359–3375, 2012.

[18] P. Kosinski and P. Maslanka, "On the definition of velocity in doubly special relativity theories," *Phys. Rev.*, vol. D68, p. 067702, 2003.

[19] S. Mignemi, "On the definition of velocity in theories with two observer-independent scales," *Phys. Lett.*, vol. A316, pp. 173–176, 2003.

[20] R. Aloisio, A. Galante, A. F. Grillo, E. Luzio, and F. Mendez, "Approaching space time through velocity in doubly special relativity," *Phys. Rev.*, vol. D70, p. 125012, 2004.

[21] G. Amelino-Camelia, F. D'Andrea, and G. Mandanici, "Group velocity in noncommutative spacetime," *JCAP*, vol. 0309, p. 006, 2003.

[22] G. Amelino-Camelia, J. R. Ellis, N. Mavromatos, D. V. Nanopoulos, and S. Sarkar, "Tests of quantum gravity from observations of gamma-ray bursts," *Nature*, vol. 393, pp. 763–765, 1998.

[23] R. Gambini and J. Pullin, "Nonstandard optics from quantum space-time," *Phys.Rev.*, vol. D59, p. 124021, 1999.

[24] G. Mandanici, "Undeformed (additive) energy conservation law in Doubly Special Relativity," *Mod. Phys. Lett.*, vol. A24, pp. 739–745, 2009.